Least Action

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***Abstract*—Explains the principle of least action**

***Index Terms*—Physics**

I. Introduction

A basic problem in calculus is to find a *critical point* of a function[1]. A critical point is a point where the function has a stationary value [2]. A sufficient condition for a critical point is $f'(x) = 0$.

An immediate generalization of this problem is to find a *critical function* of an integral. A function which takes a function as input is called a *functional*. A critical function of a functional is then a function where the integral is stationary. The value of a functional is the definite integral of a function L, called the Lagrangian.

In general we have an action functional $f \mapsto S[f]$

$$S[f] = \int_\Omega L\big(\, x, f(x), Df(x), \dots, D^k f(x) \,\big) dx$$

where

1) Ω is a region in some Euclidean space $\mathbb{R}^n$: $x \in \Omega$ and
2) f is a k-times differentiable function $f : \Omega \mapsto \mathbb{R}^n$ and
3) the Lagrangian L is a function of the appropriate number of variables.

The fundamental problem of the *calculus of variations* is to find necessary and sufficient conditions for $y = f(x)$ to make this integral stationary. Most problems are formulated in such a way as to minimize the integral.

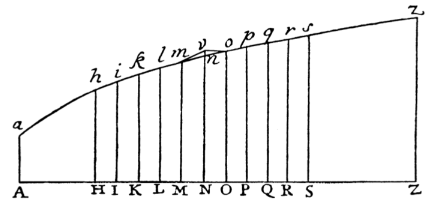
We simplify from now on the Lagrangian to $L(x, f, f')$ and assume that it is a reasonable *smooth function* in all its dependent variables.

II. Euler

In 1744 Euler published a book, [2], in which he works out a method to find a path or curve for which the integral of a given function has a stationary value. This method is known as calculus of variations and is used to solve a myriad of problems.

Euler first worked out in chapter I and II the simplest kind of these problems. Find a smooth curve $y = y(x)$ among all possible smooth curves for which the integral $\int f dx$ has a stationary value, with $f = f(x, y(x), y'(x), y''(x), \cdots)$.

We follow here Euler's approach bases on [3]. Given a curve in the plane joining the points a and z. The curve represents geometrically the analytical relation between the abscissa x and the ordinate y. Let M, N, O be three points of the interval AZ infinitely close together. We set $AM = x, AN = x', AO = x''$, and $Mm = y, Nn = y', Oo = y''$. The differential coefficient or derivative p is defined by the relation $dy = pdx$ and p' by $dy' = p'dx$. Now let $f = f(x, y(x), y'(x))$ be a function and Z the value of f at x, y, p and Z' the value at x', y', p' and so on. Then



$$\int f dx = \int_A^M f dx + Z dx + Z' dx + \cdots$$

Suppose that for the curve anz $\int f dx$ has a maximum of minimum value. Consider the curve $amvoz$ where the ordinate y' has been increased by the infinitely small quantity nv. The change in the value of $\int f dx$ must by hypothesis be zero. The only part that is affected bt varying y' is $Zdx + Z'dx$. So

$$dZ = M dx + N dy + P dp$$
$$dZ' = M' dx + N' dy' + P' dp'$$

Substituting $dx = 0, dy = 0, dy' = nv, dp = \frac{nv}{dx}, dp' = \frac{-nv}{dx}$ gives:

$$dZ = P \frac{nv}{dx}$$
$$dZ' = N' nv - P' \frac{nv}{dx}$$

The total change equals to $(dZ + dZ')dx = nv(P + N'dx - P')$. This expression must be zero. We substitute $P - P' = dP$ and $N' = N$

$$N - \frac{\mathrm{d}P}{\mathrm{d}x} = 0$$

In modern mathematics this is written as the **Euler-Lagrange equation**:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

III. Functional Derivative

To find a critical function of a functional we first define the functional derivative of:

$$S[f] = \int_a^b L(x, f, f') dx$$

with boundary conditions $f(a) = \alpha$ and $f(b) = \beta$.

Let the function f vary by adding to it another function δf that is arbitrarily small. The resulting integrand becomes: $L(x, f + \delta f, f' + \delta f')$. The differential of a functional is the

[1] a function can geometrically be represented as curve or surface

[2] a value where the function has a local extremum value, a maximum or minimum, or the function has an inflection point

part of the difference $S[f+\delta f]-S[f]$ that depends linearly on δf. At each point δf contributes to this difference. So we write for very small δf the differential:

$$\delta S[f]=\int \frac{\delta S}{\delta f}\delta f dx$$

where the quantity $\delta S/\delta f$ is called the *functional derivative* of S with respect to δf. We can write this also in the same way we are used to for the directional derivative of functions:

$$\begin{aligned} D_h S[f] &= \langle \nabla S[f]; h\rangle \\ &= \lim_{\epsilon\to 0}\frac{S[f+\epsilon h]-S[f]}{\epsilon} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}S[f+\epsilon h]\big|_{\epsilon=0} \end{aligned}$$

IV. Euler-Lagrange

We have:

$$S[f+\epsilon h]=\int_a^b L(x,f+\epsilon h,f'+\epsilon h')dx$$

Taking the derivative with respect to ϵ we first bring the derivative under the integral and then calculate the total derivative of L with respect to ϵ :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon}S[f+\epsilon h] &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_a^b L(x,f+\epsilon h,f'+\epsilon h')dx \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}\epsilon}L(x,f+\epsilon h,f'+\epsilon h')dx \\ &= \int_a^b h\frac{\partial}{\partial f}L(x,f+\epsilon h,f'+\epsilon h')+\cdots \\ &+ h'\frac{\partial}{\partial f'}L(x,f+\epsilon h,f'+\epsilon h')dx \end{aligned}$$

Setting $\epsilon=0$ gives the first variation:

$$\langle \nabla S[f]; h\rangle = \int_a^b h\frac{\partial}{\partial f}L(x,f,f')+h'\frac{\partial}{\partial f'}L(x,f,f')dx \quad (1)$$

We use integration by parts to rewrite:

$$\begin{aligned} \int_a^b h'\frac{\partial}{\partial f'}L(x,f,f')dx &= \left[h(x)\frac{\partial}{\partial f'}L(x,f,f')\right]_a^b \\ &- \int_a^b h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial f'}L(x,f,f')dx \end{aligned}$$

The boundary conditions require $h(a)=h(b)=0$ this gives:

$$\int_a^b h'\frac{\partial}{\partial f'}L(x,f,f')dx=-\int_a^b h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial f'}L(x,f,f')dx$$

Substitution into 1 gives:

$$\langle \nabla S[f]; h\rangle = \int_a^b \left(\frac{\partial}{\partial f}L(x,f,f')-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial f'}L(x,f,f')\right)hdx$$

In a function space we also have:

$$\langle \nabla S[f]; h\rangle = \int_a^b \nabla S[f]hdx$$

From which follows that the gradient of the functional S is the following function:

$$\nabla S[f]=\frac{\partial}{\partial f}L(x,f,f')-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial f'}L(x,f,f')\right)$$

For a critical function, the gradient must vanish, and the Euler-Lagrange equations follow:

$$\frac{\partial}{\partial f}L(x,f,f')-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial f'}L(x,f,f')\right)=0 \quad (2)$$

The Euler-Lagrange equation is a second order differential equation ($p=f'$):

$$E(x,f,f',f'')=\frac{\partial}{\partial f}L-\frac{\partial^2}{\partial x\partial p}L-f'\frac{\partial^2}{\partial f\partial p}L-f''\frac{\partial^2}{\partial p^2}L=0$$

V. Least Action

The trajectory, $x(t)$, of an object is determined by Newton's Law of Action. Given the initial conditions of position and velocity, the mass of the object and the force, the trajectory is determined by the vectorial differential equation:

$$\boldsymbol{F}(\boldsymbol{x},\dot{\boldsymbol{x}},t)=m\ddot{\boldsymbol{x}}$$

We can use this to solve any problem in classical mechanics. In these problems the trajectory is found step by step starting from the initial conditions by applying the Newton's Law of Motion as the state change rule. If we know the begin and end points of the trajectory could we find the curve in between without knowing the initial velocity ? Finding a curve between two points looks like a calculus of variations problem. What is being minimized along a curve defined by Newton's Law of Motion ? If we assume that the force is a conservative force, and thus depends only on the position, then we must find a Lagrangian $L(t,f(x),f(\dot{x}))$ such that the Euler-Lagrange equation is equivalent to Newton's Law of Motion. Only then we have both that the trajectory minimizes L along the curve and the curve is defined by Newton's Law of Motion. It turns out that the quantity to minimize along the curve is the difference between the kinetic energy and potential energy:

$$L(t,x,\dot{x})=T(\dot{x})-V(x)$$

This can be shown as follows:

$$\begin{aligned} \frac{\partial L}{\partial x} &= -V'(x)=F(x) \\ \frac{\partial L}{\partial \dot{x}} &= T'(\dot{x})=m\dot{x} \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) &= m\ddot{x} \end{aligned}$$

Thus,

$$F(x)-m\ddot{x}=0$$

Solving the variational problem of Least Action:

$$\min\ S=\int_a^b T(\dot{x})-V(x)dx$$

gives the curve between a and b an object follows under the influence of a force field defined by $V(x)$.

REFERENCES

[1] P. J. Olver, "Introduction to the calculus of variations," 2016.

[2] L. Euler, *method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense*, 1744. [Online]. Available: http://www.17centurymaths.com/contents/Euler'smaxmin.htm

[3] C. Fraser, "The origins of euler's variational calculus," 1993.